British Mathematical Olympiad

Round 2

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Solutions

1. Prove that if n is a positive integer, then $\frac{1}{n}$ can be written as a finite sum of reciprocals of different triangular numbers.

[A *triangular number* is one of the form $\frac{k(k+1)}{2}$ for some positive integer k.]

Solution. There are several ways to approach this problem. Some experimentation with small cases may lead to the conjecture that for every positive integer n:

$$\sum_{k=n}^{2n-1} \frac{2}{k(k+1)} = \frac{1}{n}$$

If we could prove this conjecture, we would be done because we would have expressed $\frac{1}{n}$ as the sum of reciprocals of different triangular numbers.

To prove the conjecture, first note that for any postive integer k we can write $\frac{1}{k(k+1)}$ as $\frac{1}{k} - \frac{1}{k+1}$.

Therefore,

$$\sum_{k=n}^{2n-1} \frac{2}{k(k+1)} = 2\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}\right)$$

In this sum, all the terms except the first and last cancel, so we are left with:

$$\sum_{k=n}^{2n-1} \frac{2}{k(k+1)} = 2\left(\frac{1}{n} - \frac{1}{2n}\right) = \frac{1}{n}$$

This completes this proof.

2. In an acute-angled triangle ABC with $AB < AC$, the incentre is I and the perpendicular bisector of BC meets BI at P and CI at Q. The circles BIQ and CIP meet again at X. The lines AX and BC meet at D.

Prove that D lies on the circle AQP.

Solution. Let Y be the intersection of the perpendicular bisector of BC and the line AI. We have:

$$\angle YBI = \angle YBC + \angle CBI = \angle YAC + \angle CBI = \frac{\angle CAB}{2} + \frac{\angle ABC}{2} = 90^\circ - \frac{\angle BCA}{2} = \angle YQC$$

So $BIQY$ is cyclic. Likewise, $CIPY$ is cyclic. Therefore, points X and Y are identical.

Now note that $\angle XCB = \angle XAB = \angle XAC$, so XC is tangent to the circle ADC by the converse of the Alternate Segment Theorem; hence $XC^2 = XD \cdot XA$. Moreover, in circle $CIPX$, $\angle XCI = \angle XPI = \angle CPX$ (since P is on the perpendicular bisector of BC). So XC is tangent to the circle CPQ by the converse of Alt Seg; hence $XC^2 = XP \cdot XQ$.

Combining these two equations for XC^2 we get $XD \cdot XA = XP \cdot XQ$. Thus $DAPQ$ is cyclic by the converse of the Intersecting Chords Theorem, as required.

3. An $n \times n$ chessboard consists of n^2 cells which are unit squares. Each cell is coloured black or white so that cells with a common edge are different colours. Isaac muddles up the colouring by repeatedly swapping either two complete columns or two complete rows. Elijah wants to restore the original colouring by repeatedly swapping either two complete columns or two complete rows.

In terms of n, what is the largest number of swaps that Elijah might need?

Solution. We start by proving an intermediary fact:

Lemma. After any number of operations, each pair of rows is either identically or oppositely coloured, and similarly for columns.

Indeed, suppose that we swap two rows. That preserves the property for rows automatically. Now look at the effect on an arbitrary pair of columns. If these two columns were identical before the operation, they remain identical after the operation. If they were oppositely coloured before the operation, then they are oppositely coloured after the operation. This proves the Lemma.

This means each exchange of two different columns or two different rows changes each entry in those rows or cells. Therefore the number of swaps of colour that happens in a particular cell is the sum, modulo 2, of the number of times its row has been involved in a swap of different rows and the number of times its column has been involved in a swap of different columns.

Isaac could keep track of the number of times that he swapped a given row (say the i-th row) and assign a number r_i modulo 2 to that row. He could do the same for each column to obtain column numbers c_j. Each time a cell is involved in a swap, it changes colour, so $c_i + r_j$ modulo 2 is 0 if that cell has its original colour but 1 if it has changed colour. Note that if $r_i = c_j = 0$ for all i, j, then the board has its original pattern. Also if $r_i = c_j = 1$ for all i, then it is also true that the board has its original pattern. However, that is only possible if n is even because the sum of the row numbers and the sum of the column numbers must both be 0 modulo 2 (because each swap changes either two row numbers or changes two column numbers). No other arrangements of row numbers and column numbers can yield the original colours.

First suppose that $n = 2k$ is even. The issue is to determine, for a given collection of row numbers r_i and column numbers c_j, the smallest number of swaps that will generate those numbers from either all 0s or all 1s. Determine the majority of the $2n$ row numbers and column numbers, and target that majority. the minority has size at most n and they can be "corrected". There will be an even number of target row numbers and an even number of target column numbers because there are always even numbers of 0s and even numbers of 1s among the row numbers (and among the column numbers). They can be fixed in pairs by swaps, so we can achieve constant row and column numbers using $n/2$ swaps.

Elijah may need this many swaps if Isaac has only swapped rows, and has swapped 1 and 2, 3 and 4, 5 and 6 etc (so the colour pattern is that every cell has changed colour).

If n is odd, the only arrangement of column numbers and row numbers yielding the original colour pattern is all zeroes. The total number of row numbers which are 1 is even, and the total number of column numbers which are 1 is also even (the same cannot be said for column numbers which are 0 of course). Elijah can fix the rows using at most $(n-1)/2$ swaps and similarly you can fix the columns using at most $(n-1)/2$ swaps (and no fewer will suffice), so

Elijah can fix the board using $n - 1$ swaps, and may need that many. A pessimal configuration for Elijah is if Isaac has left the bottom row and rightmost column alone, but otherwise has swapped rows 1 and 2, rows 3 and 4 etc., and has also swapped cols 1 and 2, cols 3 and 4 etc, and clearly at least $n - 1$ moves are needed to fix this. The colour pattern is that almost all cells are their correct colour; the cells in the top left $(n-1) \times (n-1)$ square are correct, but all other cells are wrong except the cell in the bottom-right corner.

So the largest number of swaps required is: $n/2$ if n is even; $n - 1$ if n is odd.

4. How many different sequences of positive integers satisfy $u_1 = 1$ and

$$u_{n+1} = \frac{(u_n^2 + u_n + 1)^{2025}}{u_{n-1}}$$

for all $n \geq 2$?

Solution. There are exactly 2026 such sequences.

On the one hand, if (u_n) is a sequence of positive integers satisfying the recurrence, then $u_3 = (u_2^2 + u_2 + 1)^{2025}$ is congruent to 1 modulo u_2. So $(u_3^2 + u_3 + 1)^{2025}$ is congruent to 3^{2025} modulo u_2. However, since u_4 is an integer, $(u_3^2 + u_3 + 1)^{2025}$ must be divisible by u_2. We deduce that $u_2 \mid 3^{2025}$. So there are 2026 possible values of u_2 (the powers of three up to 3^{2025}), and each uniquely determines the sequence (u_n).

It remains to show that all these values of u_2 lead to a sequence (u_n) consisting only of integers. Let us fix u_2 (dividing 3^{2025}) and generate the sequence (u_n) according to the recurrence relation. Suppose (for induction) that $u_1, u_2, \ldots, u_n$ are all integers for some $n \geq 4$. The recurrence relation (for $n \geq 4$) or the choice of u_2 (for $n = 3$) gives that

$$u_{n-1} \mid (1 + u_{n-2} + u_{n-2}^2)^{2025},$$

and in particular u_{n-1} and u_{n-2} are coprime. Also notice that

$$u_n u_{n-2} = (u_{n-1}^2 + u_{n-1} + 1)^{2025} \equiv 1 \pmod{u_{n-1}}.$$

Now work modulo u_{n-1}. We have

$$u_{n-2}^{4050}(u_n^2 + u_n + 1)^{2025} \equiv (u_{n-2}^2 u_n^2 + u_{n-2}^2 u_n + u_{n-2}^2)^{2025}$$

$$\equiv (1 + u_{n-2} + u_{n-2}^2)^{2025} \equiv u_{n-1} u_{n-3} \equiv 0 \pmod{u_{n-1}}.$$

Recall that u_{n-1} and u_{n-2} are coprime so u_{n-1} divides $(u_n^2 + u_n + 1)^{2025}$ and therefore u_{n+1} is an integer. By induction, the terms u_k are integers for all $k \geq 1$.

